International Journal of Theoretical Physics, Vol. 38, No. 11, 1999

Open Universes, Eternal Inflation, and the Anthropic Principle

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Received March 9, 1999

Models of open inflation with a variable density parameter Ω provide the most natural way to reconcile an open universe with inflation. The use of the anthropic principle is essential to derive observational predictions of such models. I discuss how this principle can be used in a quantitative way to determine the most probable value of Ω. Application to more general models of eternal inflation is also briefly discussed.

1. INTRODUCTION

Until recently, thinking about open universes was regarded as a waste of time because inflation predicted a flat universe. And thinking about the anthropic principle could get you into real trouble. The situation is now changing for open universes, as observations point to low values of the matter density. To meet this observational challenge, new models of inflation have been developed [1–4] in which the density parameter Ω can take a wide spectrum of values.

I think the bad reputation of the anthropic principle is also largely undeserved. If indeed we live in an open universe, then it is hard to explain the observed value of Ω without using the anthropic principle. The same applies to a universe with a nonzero cosmological constant. Here I show how the anthropic principle can be used in a quantitative way to determine the probability distribution for Ω. I first give an overview of open inflation in Section 2. Then, in Section 3, I introduce my favorite version of the anthropic principle, which I call the "principle of mediocrity." In Section 4 this principle is applied to open inflation to obtain the probability distribution

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0020-7748/99/1100-3135$16.00/0

for Ω. In Section 5, I argue that observers should not be surprised to find themselves living at an epoch when the curvature is about to dominate. Finally, in Section 6, I discuss how the principle of mediocrity can be applied to a more general class of models of eternal inflation.

2. OPEN INFLATION

2.1. One-Field Model

The simplest model of open inflation [2, 3] assumes a scalar field ϕ with a potential $V(\phi)$ of a rather special form. This potential is assumed to have a metastable false vacuum separated by a barrier from a flat, slow-roll region leading to the true vacuum. The first stage of inflation occurs when the field ϕ is stuck in the false vacuum. Occasionally ϕ tunnels through the barrier with the formation of a bubble. Then ϕ slowly rolls toward the true minimum of the potential, resulting in a second stage of inflation inside the bubble.

The process of bubble nucleation is described [5] by a compact, $O(4)$-invariant instanton which is a solution of Euclidean field equations. The nucleation probability is $\mathcal{P} \sim e^{-S_E}$, where S_E is the Euclidean action of the instanton. The evolution of a bubble after nucleation is obtained by analytically continuing the instanton to Lorentzian signature. It can be shown [5] that the bubble interior is isometric to an open Robertson–Walker universe. The homogeneity and isotropy of the bubble spacetime are ensured by the high symmetry of the instanton. Hence, in this case the horizon problem is solved not by a large amount of inflation, but by the symmetry of the bubble nucleation process.

In the space between bubbles, the rate of expansion is very high, and bubble collisions are very rare. We can therefore think of the bubbles as isolated open universes. All of these universes have the same value of Ω, which is determined by the number of e-foldings of slow-roll inflation, N. For observationally interesting values of Ω we need $N \approx 60$.

This simple model demonstrates that inflation can indeed be reconciled with a low-density universe, but it requires a substantial amount of fine tuning. The potential $V(\phi)$ is required to have a sharp barrier next to a flat, slow-roll region, which is a rather unnatural combination.

2.2. Two-Field Model

A more natural model, introduced by Linde and Mezhlumian [4], has two fields, σ and ϕ, with σ doing the tunneling and ϕ the slow roll. The potential has the form

$$V(\sigma, \phi) = V_0(\sigma) + \frac{1}{2} g\sigma^2 \phi^2 \quad (1)$$

where $V_0(\sigma)$ has a metastable false vacuum at $\sigma = 0$ and the true vacuum at $\sigma = \sigma_0$. When σ is in the false vacuum, the field ϕ is massless, while in the true vacuum it has a mass

$$m = g^{1/2} \sigma_0 \quad (2)$$

A massless scalar field in an inflating universe is subject to quantum fluctuations, which can be pictured as a random walk along the ϕ axis. After a while, all values of ϕ become equally probable.

In this model bubbles can nucleate at different values of ϕ. The nucleation probability is

$$\mathcal{P} \sim e^{-S_E(\phi)} \quad (3)$$

The number of e-foldings of the slow-roll inflation inside the bubbles is also ϕ-dependent,

$$N(\phi) \approx 2\pi G\phi^2 \quad (4)$$

Hence, one expects to have a distribution of bubbles with different values of Ω [4].

However, recent analysis by Garcia-Bellido *et al.* [6] has shown that this picture is oversimplified. The field ϕ is not homogeneous inside the bubbles. The reason is that the bubbles expand into the region of fluctuating field ϕ, and the fluctuations penetrate through the bubble wall. Mathematically, this is described by the so-called supercurvature modes. Let us denote by t_σ the time it takes the field σ to settle to its true minimum at σ_0 (here, t is the Robertson–Walker time inside a bubble). Then, on constant-time surfaces $t \sim t_\sigma$, ϕ has a Gaussian distribution

$$\mathcal{P}(\phi) \propto \exp(\phi^2/2\langle\phi^2\rangle) \quad (5)$$

with a dispersion [6]

$$\langle\phi^2\rangle \sim m^{-2} R_0^{-4} \quad (6)$$

Here, R_0 is the bubble radius at the time of nucleation. The distribution (5) is actually the same as that in Eq. (3), but their meanings are completely different. Equation (3) gives the probability that a bubble nucleates with a given value of ϕ, while (5) gives the probability distribution for ϕ inside a single bubble.

Only regions where ϕ is greater than the Planck mass m_p are going to inflate, and the amount of inflation will differ from one region to another, according to Eq. (4). Hence, each bubble will contain an infinite number of regions with different values of Ω. Garcia-Bellido *et al.* called this picture "quasiopen inflation."

The correlation length of ϕ inside the bubbles at $t \sim t_\sigma$ is $\xi \sim R_c / H_F^2 m^2 R_0^4$, where H_F is the expansion rate in the false vacuum and R_c is the curvature radius of the hypersurfaces $t \sim t_\sigma$. This correlation length sets the length scale of variation of Ω and must be much greater than the comoving size of the presently observable universe (by a factor of at least 10^7, in order that the microwave background anisotropies do not get unacceptably large). This can be enforced by a suitable choice of parameters in the potential (1).

Clearly, in this type of model, the value of Ω cannot be predicted with certainty. One can only try to determine the probability distribution for Ω.

3. PRINCIPLE OF MEDIOCRITY

Each of the bubbles will be inhabited by an infinite number of civilizations which will generally measure different values of Ω. In some regions Ω will be too low for any galaxies to form. The probability for measuring such values of Ω should be set equal to zero, since there will be nobody there to observe them. When people talk about the anthropic principle, they usually mean this anthropic constraint (see, e.g., ref. 7). However, I suggest that we use a much more quantitative version [8].

My suggestion is that the probability $\mathcal{P}(\Omega)d\Omega$ for Ω to be in the interval $d\Omega$ should be set proportional to the number of civilizations which will measure Ω in that interval. Assuming that we are a typical civilization, we can expect to observe a value of Ω near the maximum of $\mathcal{P}(\Omega)$. This version of the anthropic principle, which I called the principle of mediocrity [8], is an extension of the Copernican principle. The Copernican principle asserts that our position in space is not special, while the principle of mediocrity asserts that the values of the cosmological parameters we are going to measure are not special either.[2]

In order to apply the principle of mediocrity to our models, we will have to compare the number of civilizations in parts of the universe with different values of Ω. Of course, we cannot calculate the number of civilizations. However, since the value of Ω does not affect the physical processes involved in the evolution of life, this number must be proportional to the number of habitable stars or, as a rough approximation, to the number of galaxies. Hence, we shall set the probability for us to observe a certain value of Ω to be proportional to the number of galaxies formed in parts of the universe where Ω takes the specified value.

[2] A very similar approach was used by Carter [9], Leslie [9], and Gott [11] to estimate the expected lifetime of our species. Gott also applied it to estimate the lifetimes of various political and economic structures, including the journal *Nature* where his article was published.

The principle of mediocrity was applied to calculate the probability distribution for Ω in an earlier paper [12], which assumed the old picture of homogeneous open universes inside bubbles. A serious difficulty encountered in that calculation was that open universes inside the bubbles have infinite volume and contain an infinite number of galaxies. Thus, to find the relative probability for different values of Ω, one had to compare infinities, which is an inherently ambiguous task. This problem was addressed in ref. 12 by introducing a cutoff and counting only galaxies formed prior to the cutoff. Although the cutoff procedure employed in ref. 12 has some nice properties, it is not unique, and the resulting probability distribution is sensitive to the choice of cutoff [13]. This cutoff dependence, which also appears in other models of eternal inflation [13, 14], has led some authors to doubt that a meaningful definition of probabilities in such models is even in principle possible [13, 15].

However, this pessimistic conclusion may have been premature. According to the quasiopen picture, Ω takes all its possible values within each bubble. Since all bubbles are statistically equivalent, it is sufficient to consider a single bubble. Moreover, we can restrict ourselves to a finite (but very large) comoving volume within that bubble, provided that its size is much greater than the characteristic scale of variation of Ω. Thus, we no longer need to compare infinities, and the problem becomes well defined.

In the next two sections I shall report on a recent paper I wrote with Jaume Garriga and Takahiro Tanaka [16] where we extend the work of ref. 12 in two important respects. First, we use the quasiopen picture (which actually makes the calculation much simpler) and second, we give a much more careful treatment of the astrophysical aspects of galaxy formation.

The principle of mediocrity has also been applied to other cosmological parameters, e.g., the cosmological constant [17–19] and the amplitude of density fluctuations [20].

4. CALCULATION OF $\mathcal{P}(\Omega)$

A great simplification introduced by the quasiopen picture is that Ω takes all its possible values within each bubble. Since all bubbles are statistically equivalent, it is sufficient to calculate $\mathcal{P}(\Omega)$ for a single bubble.

The distribution $\mathcal{P}(\Omega)$ can be expressed as

$$\mathcal{P}(\Omega) \propto \mathcal{P}(\phi) e^{3N(\phi)} \nu(\Omega) \left| \frac{d\phi}{d\Omega} \right| \tag{7}$$

Here, $\mathcal{P}(\phi)$ is the probability distribution for ϕ, Eq. (5), and the next factor accounts for the fact that regions starting out with different values of ϕ will inflate by a different amount; $e^{3N(\phi)}$ is the volume enhancement factor. The "anthropic factor" $\nu(\Omega)$ is proportional to the number of galaxies formed per unit volume. More precisely, it is the fraction of galactic-scale volumes which eventually collapse to form galaxies. Finally, the last factor in (7) is the Jacobian transforming from ϕ to Ω via the relation [2]

$$H_*^2 \, e^{2N(\phi)} \approx \frac{T_*^2}{T_{eq}T_{CMB}} \frac{\Omega}{1 - \Omega} \tag{8}$$

Here, H_* and T_* are the expansion rate and the temperature at the end of inflation, T_{eq} is the temperature at equal matter and radiation densities, and T_{CMB} is the temperature at the time when the value of Ω is specified.

The Gaussian distribution $\mathcal{P}(\phi)$ is peaked at $\phi = 0$, which corresponds to $\Omega = 0$, while the volume factor favors large values of ϕ and pushes Ω toward $\Omega = 1$. The effect of $\nu(\Omega)$ can be understood if we note that the growth of density fluctuations in an open universe terminates at a redshift $1 + z \sim \Omega^{-1}$. If Ω is too low, galaxy formation is suppressed. Hence, $\nu(\Omega) \to 0$ as $\Omega \to 0$.

An interesting situation arises when $\mathcal{P}(\phi)$ dominates over the volume factor. In this case, the effect of the anthropic factor $\nu(\Omega)$ is to shift the peak of the distribution from $\Omega = 0$ to a nonzero value of Ω.

In order to calculate $\nu(\Omega)$, one needs to make some assumptions about the nature of cosmological density fluctuations. We shall assume Gaussian fluctuations characterized by a dispersion σ_{rec} on the comoving galactic scale at the time of recombination. The choice of reference time here is unimportant, as long as it is much earlier than the time of curvature domination, so that one can reasonably assume that σ_{rec} is independent of Ω. For the galactic scale we choose the comoving scale of 1 Mpc at present.

In an open universe with a density parameter Ω_{rec} at t_{rec}, the dispersion of density fluctuation in the asymptotic future, σ_∞, is greater than σ_{rec} only by a finite factor,

$$\frac{\sigma_\infty}{\sigma_{rec}} = \frac{5}{2} \frac{\Omega_{rec}}{1 - \Omega_{rec}} = \frac{5\, T_{rec}}{2T_{CMB}} \frac{\Omega}{1 - \Omega} \tag{9}$$

where in the last step I used the fact that

$$T(1 - \Omega)/\Omega = \text{const} \tag{10}$$

during the matter era. We can now use the Press–Schechter formalism to determine $\nu(\Omega)$. Galaxies will form in regions where the linearized density contrast δ exceeds the critical value $\delta_c \approx 1.7$. Hence, $\nu(\Omega)$ is given by the integral of the Gaussian distribution over $\delta > \delta_c$, that is, by the error function [21]

$$\nu(\Omega) = \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_\infty}\right) = \mathrm{erfc}\left(\kappa \frac{1-\Omega}{\Omega}\right) \tag{11}$$

where

$$\kappa = \frac{2}{5}\frac{T_{\mathrm{CMB}}}{T_{\mathrm{rec}}}\frac{\delta_c}{\sqrt{2}\sigma_{\mathrm{rec}}} \tag{12}$$

In principle, the value of σ_{rec} can be determined from the fundamental theory of density fluctuations. Until such theory is known, we can, in practice, adjust σ_{rec} to fit the CMB observations. Our ability to do so is limited, however, by the fact that the value of σ_{rec} inferred from observations depends on the Hubble "constant" H_0 and the density parameter Ω_0 in our part of the universe, which are not very well determined. With this uncertainty [16],

$$\kappa = 0.1 \pm 0.05 \tag{13}$$

Combining Eqs. (5)–(8) and (11), we can now write the final form of the probability distribution for Ω,

$$\frac{d\mathcal{P}}{d\ln y} \propto y^{3(\mu-1/2)}\,\mathrm{erfc}(y) \tag{14}$$

Here,

$$y = \kappa\frac{1-\Omega}{\Omega} \tag{15}$$

and

$$\mu = \frac{m_p^2}{24\pi\langle\phi^2\rangle} \sim m_p^2\, m^2\, R_0^4 \tag{16}$$

Note that the variable y defined in (15) does not depend on the temperature T_{CMB} at which Ω is measured, because of Eq. (10).

For $y > 1$, the error function can be approximated as

$$\mathrm{erfc}(y) \approx \frac{1}{\sqrt{2}y}\, e^{-y^2} \tag{17}$$

and the value of Ω at the peak of the distribution (14) can be expressed analytically,

$$\kappa\left(\frac{1-\Omega}{\Omega}\right)_{\mathrm{peak}} \approx \left(\frac{3}{2}\mu - \frac{5}{4}\right)^{1/2} \tag{18}$$

The peak is rather broad, with a width

$$\Delta\left(\frac{1-\Omega}{\Omega}\right) \sim 5 \tag{19}$$

Interesting values of Ω_{peak}, which are not too close to either 0 or 1, are obtained in models with $\mu \sim 1$ (which can be easily constructed). Further details of the calculations and the results can be found in ref. 16.

The moral of this analysis is that, given a particle physics model, the probability distribution for Ω can be unambiguously calculated from first principles.

5. THE COSMIC AGE COINCIDENCE

The usual objection against models with $\Omega < 1$ is that it is hard to explain why we happen to live at the epoch when the curvature is about to dominate. That is, why

$$t_0 \sim t_c \tag{20}$$

where t_0 is the present time and t_c is the time of curvature domination. Observers at $t << t_c$ would find $\Omega \approx 1$, while observers at $t >> t_c$ would find $\Omega << 1$. It appears that one needs to be lucky to live at a time when Ω differs substantially from both 0 and 1. Here I am going to argue that the coincidence (20) is not as surprising as it may first seem [16].

From the analysis in the preceding section, we can expect to have

$$t_c \sim t_G \tag{21}$$

where t_G is the time of galaxy formation. Without the anthropic factor $\nu(\Omega)$, the probability distribution for Ω is peaked at $\Omega = 0$, which corresponds to $t_c \to 0$. The role of $\nu(\Omega)$ is to push the peak to values of Ω such that the curvature domination occurs soon after galaxies are formed, so that $t_c \sim t_G$.

Recall Dicke's observation [22] that the present time t_0 is likely to be comparable to the lifetime of a typical main sequence star, $t_0 \sim t_\star \sim 10^{10}$ years. Moreover, observationally the epoch of structure formation, when giant galaxies were assembled, is at $z_G \sim 1$–3, or $t_G \sim 10^9$–10^{10} years. Since, $t_G \sim t_\star \sim t_0$, it follows from (21) that $t_c \sim t_0$.

The above argument is based on the coincidence

$$t_G \sim t_\star \tag{22}$$

which cannot be explained in the framework of our model. The time of galaxy formation t_G depends on the amplitude of the cosmological density fluctuations, while the stellar lifetime $t_\star$ is determined by the fundamental constants. In our model, the only variable is t_c, while t_G and $t_\star$ remain fixed. It is conceivable that the coincidence (22) may find some kind of anthropic explanation in more general models where some other "constants" are allowed to vary.

6. MORE GENERAL MODELS

The approach to calculating probabilities outlined in Sections 3 and 4 can be extended to more general models in which "constants of nature" and cosmological parameters other than Ω can take different values in different parts of the universe. Let us consider models with a single inflaton field ϕ and a potential $V(\phi)$ having no false vacua. (Such models predict $\Omega \approx 1$.) A remarkable property of inflation is that generically it never ends in the entire universe. The evolution of the inflaton field ϕ is influenced by quantum fluctuation, and as a result thermalization does not occur simultaneously in different parts of the universe. The dynamics of ϕ can be pictured as a random walk superimposed on the classical slow roll. On small scales, the fluctuations in the thermalization time give rise to a spectrum of small density fluctuations, but on large scales they make the universe extremely inhomogeneous. In most of the models one finds that at any time there are parts of the universe that are still inflating and that the total volume of the inflating regions is growing with time. This picture is often referred to as "stochastic inflation" or "eternal inflation" [23, 24].

Eternally inflating universes may contain thermalized regions characterized by different values of the constants of nature and of the cosmological parameters. For example, the inflaton potential $V(\phi)$ may have several minima corresponding to different low-energy physics or to different values of the cosmological constant. A more interesting possibility is that the "constants" are related to some slowly varying fields and take values in a continuous range. Just like the inflaton field ϕ, the fields χ_j ($j = 1, \ldots, n$) associated with the "constants" are subject to quantum fluctuations during inflation, and different regions of the universe thermalize with different values of χ_j. According to the principle of mediocrity, the probability $\mathcal{P}(\chi)d^n\chi$ for χ_j to be in the intervals $d\chi_j$ is proportional to the number of civilizations which will measure χ_j in that interval.

As before, the problem of calculating $\mathcal{P}(\chi)$ can be split into two parts. The probability for us to observe certain values of χ_j is proportional to the volume of the regions where χ_j take specified values and to the density of galaxies in those regions. It is convenient to consider comoving regions and measure their volume at the time of thermalization. Then we can write

$$\mathcal{P}(\chi) \propto \nu(\chi)\, \mathcal{P}_*(d\chi) \tag{23}$$

Here, $\mathcal{P}_*(\chi)\, d^n\chi$ is proportional to the volume of thermalized regions where

χ_j take values in the intervals $d\chi_j$, and $\nu(\chi)$ is the number of galaxies that form per unit thermalized volume.

The factor $\mathcal{P}_*(\chi)$ in Eq. (1) is the probability distribution of the fields χ_j on the thermalization hypersurface Σ_* which separates inflating and thermalized spacetime regions. It is a three-dimensional spacelike surface which plays the role of the "big bang" for the thermalized regions. In the case of several discrete vacua, Σ_* consists of a number of disconnected pieces, each connected component corresponding to thermalization into a single vacuum.[3] Each connected piece of Σ_* has an infinite volume. (The situation here is similar to open inflation, where thermalized regions inside the bubbles have the form of infinite, open Robertson–Walker universes). In order to determine the relative probability of different vacua, one has to compare the infinite volumes of the corresponding components of Σ_*, which is an inherently ambiguous task. This is the same ambiguity that we encountered in Section 3 in the case of open inflation.

Now, the key observation [26] is that the situation may be greatly improved in the case of continuous fields χ_j varying in a finite range, $0 \leq \chi_j \leq \Delta_j$. In this case, each connected part of Σ_* is still infinite, but now different parts are not characterized by different values of χ_j. On the contrary, χ_j run over all the range of their values on each connected part. Since the inflationary dynamics of the fields χ_j has a stochastic nature, the distributions of χ_j on different components of Σ_* should be statistically equivalent. It is therefore sufficient to consider a single connected component. Moreover, since χ_j have a finite range, they will run over all of their values many times on any sufficiently large part of Σ_*. Hence, there is no need to deal with infinite hypersurfaces. The probability distribution $\mathcal{P}_*(\chi)$ can be determined by examining a large, finite piece of Σ_*.

This argument can be extended to fields with an infinite range of variation, provided that the probability distributions of χ_j are concentrated within a finite range, with a negligible probability of finding χ_j very far away from that range.

Models with a finite range of χ_j are not difficult to construct. For example, χ_j, could play the role of angular variables, with the inflaton field ϕ being the radial variable in the field space. Another attractive possibility is a σ-model-type theory in which both χ_j and ϕ take values on a compact manifold. Some examples are given in ref. 26.

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